

Individual Round — Arithmetic

- (1) What is two-thirds times three-quarters ?
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$1/2$

- (2) True or False: If the average of twenty exam scores is 75.65% (and no one actually had a score of 75.65%) there must be ten people who scored higher and ten who scored lower.
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False. For example, there could have been 19 people scoring 76 and 1 person scoring 69.

- (3) At a summer camp, there are 20 four year-olds, 15 six year-olds, 10 eight year-olds and 5 fifteen year-olds. What is the average age of the campers?
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The average is $(20 \cdot 4 + 15 \cdot 6 + 10 \cdot 8 + 5 \cdot 15)/50$, which comes to 6.5 years.

- (4) 2011 is written with exactly 3 different digits. How many years from 1900 to 2011 (inclusive) have exactly 3 different digits?
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Note that the years in the given range that start with a 2 and have 3 digits are 2001, 2003, 2004, 2005, 2006, 2007, 2008, 2009, and 2011 itself. Thus there are 9 such years in the 21st century. In the 20th century, all the years begin with 19. There must be a digit that is different from 1 and 9 which we'll call d . There are 8 possibilities for d : 0, 2, 3, 4, 5, 6, 7, and 8. It is possible to list the potential forms of the years in terms of d : 19dd, 191d, 199d, 19d1 and 19d9. This gives a total of $(5 \cdot 8) + 9 = 49$ years in the given range that have exactly 3 distinct digits.

Individual Round — Algebra

- (1) Solve the linear equation $5x + 2 = 17$.
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$$x = 3$$

- (2) Consecutive perfect squares always differ by an odd number. For example $6^2 - 5^2 = 11$. Which two consecutive squares differ by 2011?
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Translating the word problem into an algebraic formulation gives

$$(x + 1)^2 - x^2 = 2011$$

so

$$(x^2 + 2x + 1) - x^2 = 2011$$

and finally

$$2x + 1 = 2011.$$

So $x = 1005$ and the two squares are $1005^2 = 1010025$ and $1006^2 = 1012036$.

- (3) If the graphs of the lines whose equations are $3x - 5y + 2 = 0$ and $2x + Ay - 11 = 0$ are perpendicular, find the value of A .
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By solving both equations for y (putting them in $y = mx + b$ form) we can determine that their slopes are $3/5$ and $-2/A$. These numbers must be negative reciprocals if the lines are to be perpendicular which leads us to the equation $-2/A = -1/(3/5)$ which has solution $A = 6/5$.

- (4) Find all possible solutions of $|x^2 - 12x + 31| = 4$.
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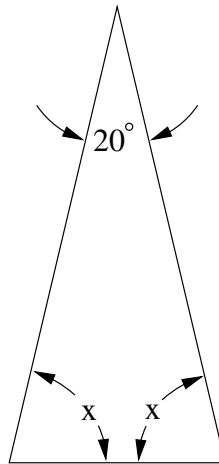
Because of the absolute values, we either have

$$x^2 - 12x + 31 = 4 \quad \text{or} \quad x^2 - 12x + 31 = -4.$$

Both alternatives give rise to quadratic equations that can be solved by factoring. The solutions are 9, 3, 5 and 7.

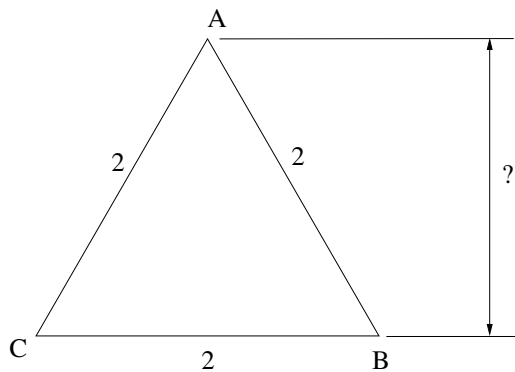
Individual Round — Geometry

- (1) What is the measure of the angles marked x in the following isosceles triangle?
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$$x = 80^\circ$$

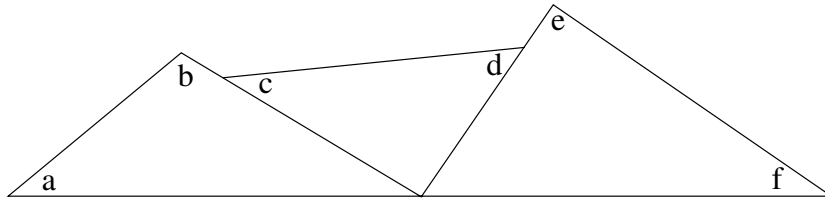
- (2) The triangle $\triangle ABC$ is an equilateral triangle – all of its sides are of length 2. What is its altitude?
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By drawing in an altitude of this triangle you will create a right triangle with hypotenuse 2 and one leg (the base) of length 1. The other leg (the altitude) can be found using the Pythagorean formula: $1^2 + y^2 = 2^2$ which leads to $y = \sqrt{3}$

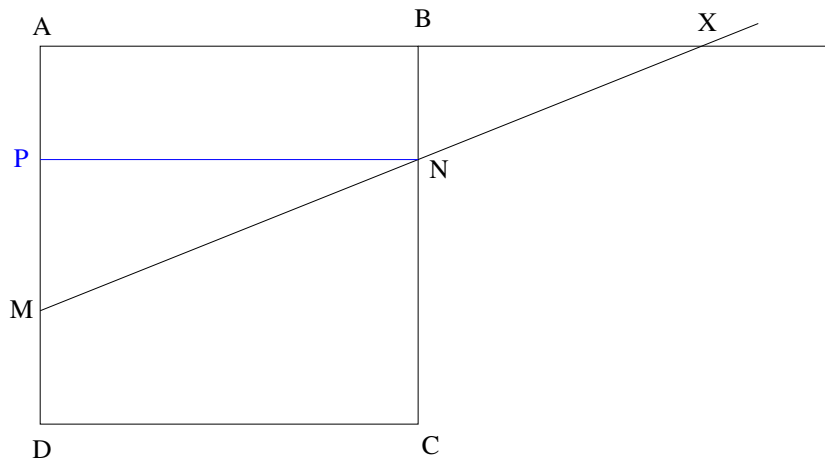
- (3) Find the value in degrees of the sum of the six angles indicated in the diagram.

$$a + b + c + d + e + f = \underline{\hspace{2cm}}$$



The three angles that are *not* labeled add up to 180° and the total of all 9 angles in the three triangles is $3 \cdot 180^\circ = 540^\circ$. The labeled angles must add up to 360° .

- (4) ABCD is a 10cm by 10cm square. Points M and N are placed so that $\overline{BN} = \overline{MD} = 3$. The line through M and N is extended until it intersects the line through A and B. The point of intersection is labeled X. What is the length of segment BX?



Construct point P on segment AM so that it lies 3cm below A . Note that $\triangle MPN$ and $\triangle NBX$ are similar. It follows that $x/3 = 10/4$ (where x is the unknown length of segment BX) and this leads us to an answer of 7.5.

Team Round — CAPT

As part of a promotion, a radio station is giving away 100 smart phones to the first 100 people in line at a concert. To keep it interesting, they decided that 20 of the phones will have a \$100 bill slipped in behind the battery, so they will appear to be broken.

- (1) Suppose you are the first person selected to receive a phone. What is the probability that you get a “broken” phone?

20/100 or 1/5.

- (2) Suppose you are the 50th person to receive a phone and you know that 8 people who went before you have found \$100 bills in their phones. What is the probability now that you get a “broken” phone?

Your phone is selected from a group of 51 phones (note that 49 people went before you) and there are 12 \$100 bills still unaccounted for, so the probability is 12/51.

- (3) After all the phones have been given out a friend tells you that both she and her cousin got phones. (For this question, assume that you have no further information – you don’t know where they were in line, etc.) What is the probability that at least one of them got a phone with a \$100 bill?

Consider instead the probability that *neither* of them got a \$100 bill, which is $\frac{80}{100} \cdot \frac{79}{99} = 0.63838$. The probability we seek is the complement of this one, which is 0.3616 .

An acceptable approximation is $1 - (.8)^2 = 0.36$.

- (4) You went to the concert as part of a friend’s birthday party, there was a total of 6 of you, and naturally you were all near the front of the line so all six got phones. What is the probability that *none* of you got a \$100 bill?

$\frac{80}{100} \cdot \frac{79}{99} \cdot \frac{78}{98} \cdot \frac{77}{97} \cdot \frac{76}{96} \cdot \frac{75}{95} \approx 0.252$.

Again, it is reasonable to ignore the slightly better odds one has if those who have gone before *didn’t* get a \$100 bill. So $(0.8)^6 = 0.262144$ is an acceptable answer.

Team Round — General

- (1) Given that $5^n + 5^n + 5^n + 5^n + 5^n = 5^{25}$, find n .
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$5^n + 5^n + 5^n + 5^n + 5^n = 5 \cdot 5^n = 5^{n+1}$. So we really just need to solve $n + 1 = 25$. Clearly the answer is $n = 24$.

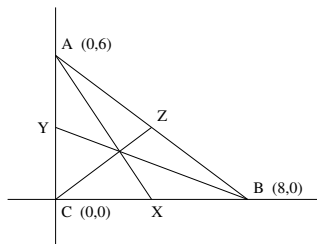
- (2) Two circles are inscribed within a larger circle of diameter 10cm so that they are tangent to one another and to the outer circle, and so that their centers are collinear. What is the sum of the circumferences of the 3 circles?
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It is strange, but true, that the answer does not depend on what the actual diameters of the smaller circles are. If we call their diameters x and y (respectively) then their circumferences are πx and πy . Notice that $\pi x + \pi y = \pi(x + y) = 10\pi$, which is also the circumference of the largest circle. Thus, the total is 20π .

- (3) How many zeros are there at the end of $8^{26} \cdot 5^{79}$?
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$8^{26} \cdot 5^{79} = (2^3)^{26} \cdot 5^{79} = 2^{78} \cdot 5^{79} = 5 \cdot 10^{78}$ so there are 78 zeros.

- (4) Triangle $\triangle ABC$ has its vertices at $(0, 6)$, $(8, 0)$ and $(0, 0)$. The points X , Y and Z are the midpoints of segments BC , AC and AB respectively. The line segments AX , BY and CZ are known as the *medians* of the triangle. It is a remarkable fact that the medians all intersect in a single point. What are the coordinates of this point of intersection of the medians (known as the *centroid* of the triangle)?
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Notice that point X has coordinates $(4, 0)$ and that point Y has coordinates $(0, 3)$. So we have the coordinates for two points on each of the lines AX and BY . This allows us to determine the slopes of these lines and their y -intercepts are evident, so their equations are

$$y = -\frac{3}{8}x + 3 \quad \text{and} \quad y = -\frac{3}{2}x + 6$$

This gives us a system of 2 equations in 2 unknowns which has a unique solution: $x = 2\frac{2}{3}$ and $y = 2$.