New Haven High School Math Contest — 2007
Individual Round — Arithmetic

(1) What is 75% of 2/3?

$$75\% \cdot 2/3 = \frac{75}{100} \cdot \frac{2}{3} = \frac{3}{4} \cdot \frac{2}{3} = \frac{6}{12} = 1/2$$

(2) The product of two distinct positive integers is 42. What is the smallest possible sum of these integers?

13. The factors are either 1 & 42 or 2 & 21 or 3 & 14 or 6 & 7. The last set of factors (6 & 7) give the smallest sum which is 13.

(3) What is the value of the following alternating sum?

$$1 - 2 + 3 - 4 + 5 - 6 + \ldots + 2005 - 2006 + 2007$$

Each pair of terms amounts to subtracting 1. By the time we reach 2006, we have subtracted one 1003 times (so at the next to last stage we have -1003). When we add the final 2007 we get $+1004$.

(4) How many four digit whole numbers have an odd digit in the thousands place, an even digit in the hundreds place, and have all four digits different?

There are 5 choices for the thousands place (1, 3, 5, 7 or 9). There are also 5 choices for the hundreds place (0, 2, 4, 6 or 8 (and yes, 0 *is* even.) For the tens place we only have 8 possibilities because we can't reuse the digits that we put in the hundreds and thousands places. Similarly, we have 7 possible choices for the digit in the ones place. The final answer is $5 \cdot 5 \cdot 8 \cdot 7 = 1400$.

New Haven High School Math Contest — 2007
Individual Round — Algebra

(1) Find a value for x that satisfies the equation $3x + 2 = 10 - x$.

$$3x + 2 = 10 - x$$

subtract 2 from both sides

$$3x = 8 - x$$

add x to both sides

$$4x = 8$$

divide both sides by 4

$$x = 2$$

(2) Find both values of x that satisfy $|1 - x| = 6$.

The solutions to the absolute value equation occur when either $1 - x = 6$ or $1 - x = -6$. The first of these gives us $x = -5$ and the other leads us to $x = 7$.

(3) For what values of x does the reciprocal of $x + 1$ equal $x - 1$?

Translating the words into an equation results in

$$\frac{1}{x + 1} = x - 1.$$

Since $x = -1$ is clearly not a solution to the problem, we can assume $x \neq -1$ and so we can multiply both sides by $x + 1$ to obtain

$$1 = (x + 1)(x - 1) = x^2 - 1.$$

This simplifies to $x^2 = 2$ and the solutions are obvious from there: $x = \pm\sqrt{2}$.

(4) If $f(x) = x^2 - 3x - 5$, what are the values of k such that $f(k) = k$?

Using the given formula for $f(x)$ we can rewrite $f(k) = k$ as $k^2 - 3k - 5 = k$. Subtract k from both sides to get $k^2 - 4k - 5 = 0$ which can be factored into $(k - 5)(k + 1) = 0$. The solutions can be seen in the factored form: $k = 5$ and $k = -1$.

New Haven High School Math Contest — 2007
Individual Round — Geometry

(1) Recall that a triangle whose sides are 3, 4 and 5 inches long is a right triangle. What is its area?

You need the area formula for a triangle, $A = bh/2$. Use $b = 3$ and $h = 4$ (or vice versa) to find the area: 6.

(2) What fraction of the area of the rectangle illustrated below is shaded?
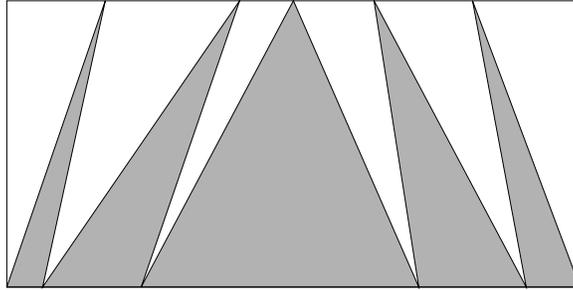
Half. All the triangles have the same altitude and the total of the bases of the white triangles is the same as the total of the bases of the gray triangles.

(3) A wooden cube has an edge length of 4 inches. The cube is painted red all over and then cut up into 64 little 1 by 1 by 1 cubes. How many of these little cubes have exactly one face painted red?

Each face of the cube will have 4 little cubes in its center that have only one face painted. So there are 24 such little cubes.

(4) In the figure below, AC is an arc of a circle centered at O. The line segments $\overline{OA}$ and $\overline{AB}$ are perpendicular and both have length 12 centimeters. What is the area of the shaded region?
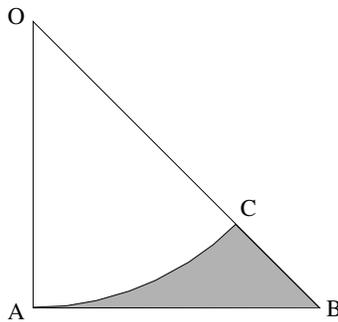
The right triangle $\triangle OAB$ has area $12 \cdot 12/2 = 72$. The unshaded circular sector is one eighth of a circle of radius 12, so its area is $\frac{1}{8}\pi 12^2$ or 18π. The exact answer is $72 - 18\pi$. A decimal approximation is 15.451.

Team Round — CAPT

JoAnne has landed a great job which begins right after graduation. She will be commuting 15,000 miles each year, so she needs to buy a fuel efficient car for commuting. She is considering three different models: the Toyota Prius Hybrid, the Honda Civic Hybrid, and the Honda Civic DX (a traditional 4 cylinder engine). For the purpose of making comparisons JoAnne assumes that she will keep her car for five years, so she needs to compute the total cost of purchasing and operating each vehicle over a five year period.

The purchase prices and estimated gas mileages of these three vehicles are given in the following table.

Make/Model	Price	Mileage
Toyota Prius Hybrid	$22,175	55 mpg
Honda Civic Hybrid	$22,600	50 mpg
Honda Civic DX	$18,710	35 mpg

(1) (1 pt) Assuming gasoline costs $3.00 per gallon, which of the two hybrid models will be cheaper to purchase and operate over a 5 year period?

Notice that the Prius is both cheaper *and* gets better gas mileage than the Honda. You don't need to do any calculations to see that the Prius is a better deal. This doesn't mean the NHHSMC endorses Toyota — for instance the Honda has an impressive warranty...

(2) (2 pts) JoAnne decides she doesn't like the appearance of the Honda Civic Hybrid, so she just needs to decide between the Prius and the Civic DX. If gas is cheap enough, she knows that the lower purchase price of the Civic DX should make it cost less to purchase and operate over a five year period than the Prius. On the other hand, if gasoline becomes very expensive the Prius will be cheaper. At what price per gallon for gasoline does the Prius become cheaper than the Civic DX? (We will refer to this as the "break even price.")

Actually, we here at the NHHSMC *like* the looks of the Honda Hybrid ...

Let x represent the price per gallon of gasoline. Note that 15,000 miles a year for five years is 75,000 miles. The total cost to purchase and drive the Prius for 5 years will be $22175 + x \cdot 75000/55$. The total cost to purchase and drive the Civic DX for 5 years will be $18710 + x \cdot 75000/35$. We set these total costs equal to one another and solve for x:

$$22175 + x \cdot 75000/55 = 18710 + x \cdot 75000/35$$

We'll skip all the algebra, but in the end you shuold get

$$x = \frac{3465 \cdot 77}{15000 \cdot 4}$$

That's approximately $4.44\frac{7}{10}$ per gallon!

(3) (1 pt) If we change our assumptions so that JoAnne needs to drive 25,000 miles per year, what is the break-even gasoline price?

This is just like the last problem except we are driving a total of 125,000 miles in five years so we have to solve

$$22175 + x \cdot \frac{125000}{55} = 18710 + x \cdot \frac{125000}{35}.$$

The answer is approximately $2.66\frac{8}{10}$ per gallon.

(4) (3 pts) Express the break-even gasoline price as a function of the number of miles driven per year.

Let y represent the number of miles driven per year. Replace the numbers 15,000 and 25,000 that appeared in the last two problems with y. Our equation is now

$$22175 + x \cdot \frac{5y}{55} = 18710 + x \cdot \frac{5y}{35}.$$

We need to solve this for x. The final answer is

$$x = 66701.25 \cdot \frac{1}{y}.$$

As a check you can verify that this formula gives our previous two answers when we substitute $y = 15000$ and 25000 respectively.

(5) (3 pts) JoAnne realizes she will need to drive some number of miles each year in addition to her commute. Assume that gas stays at $3.00 per gallon. What would be the total number of miles (over the entire five year period) that would give the same total costs for the Prius and the Civic DX?

We could return to the original formula and solve from there, but it is much easier to use the relationship we just deduced between x and y and see what value for y corresponds to $x = 3$.

Just solve the equation $3 = 66701.25 \cdot \frac{1}{y}$ for y.

Thus $y = \frac{66701.25}{3} = 22,233.75$.

That's only about 7,000 miles per year of additional driving beyond her commute – which is pretty reasonable. The two cars actually have very similar costs when we only consider purchase price and gas costs. There are other factors like maintenance costs, insurance and annual taxes that would need to be considered to make this decision.

Team Round — General

(1) What is the sum of all the odd numbers from 1 to 2007?

$$1 + 3 + 5 + \ldots + 2005 + 2007 =$$

Look at a few of the sums of the first several odd numbers.

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16$$

You should be able to recognize the pattern, the sum of the first n odd numbers is n^2. Note that 2007 is the 1004th odd number so

$$1 + 3 + 5 + \ldots + 2005 + 2007 = 1004^2 = 1008016.$$

(2) How many integers between 0 and 1,000 are multiples of 4 and **do not** contain any of the digits 6, 7, 8, 9 or 0?

It's easy to list the numbers less than 100 that satisfy given conditions:

$$4, \quad 12, \quad 24, \quad 32, \quad 44, \text{ and } 52.$$

We can put the digits $\{1, 2, 3, 4, \text{and} 5\}$ in front of all of these except 4 to get a three digit number satisfying the given conditions. Thus there are $1 + 5 + 25 = 31$ numbers of the desired form.

(3) Suppose a function $f(x)$ has the property that $f(1/x) - 2 \cdot f(x) = x$ for all values of x, other than 0. What is $f(2)$?

By substituting $x = 2$ and $x = 1/2$ we obtain two independent equations.

$$f(1/2) - 2 \cdot f(2) = 2$$
$$f(2) - 2 \cdot f(1/2) = 1/2$$

It may make the algebra easier to follow if we do some renaming. Let $a = f(2)$ and $b = f(1/2)$. We'll need to find a.

Using the new variables, the system of equations now looks like

$$b - 2a = 2$$
$$a - 2b = 1/2.$$

Double the first equation to obtain

$$2b - 4a = 4$$
$$a - 2b = 1/2$$

now add the two equations to get $-3a = 4\frac{1}{2}$ so $a = -1\frac{1}{2}$.

(4) A *Pythagorean triple* is an ordered triple of positive integers which form the sides of a right triangle. (For example, $(3, 4, 5)$ is a Pythagorean triple.) Determine a Pythagorean triple such that one of the integers is 2007.

This is easy if you happen to notice that 2007 is divisible by 3 (the sum of the digits is 9 which is divisible by 3). Just scale the $(3, 4, 5)$ triangle by a factor of $669 = \frac{2007}{3}$ to obtain the Pythagorean triple $(2007, 2676, 3345)$.

There are other possible solutions such as $(2007, 2014024, 2014025)$.