Individual Round — Arithmetic

(1) Recall that the *factorial* of a number is denoted using an exclamation point, its value is the product of all the integers from 1 up to that number (for example, $4! = 1 \cdot 2 \cdot 3 \cdot 4$).

Solve for N if

$$6! \cdot 7! = N!.$$

$$\begin{aligned} 6! \cdot 7! &= (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \\ &= (2 \cdot 3 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \\ &= (2 \cdot 5 \cdot 3 \cdot 3 \cdot 4 \cdot 2 \cdot 1) \cdot (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \\ &= (10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \\ &= 10! \end{aligned}$$

(2) Find the exact value of $(111,111,111)^2$.

Think about the process of finding the number via long multiplication. There will be no carries. The answer is 12345678987654321.

(3) How many positive factors does 900 have?

Since 900 is $2^2 \cdot 3^2 \cdot 5^2$, any factor of 900 will involve only those primes. The exponents will be no greater than 2 (on each prime) so there are three choices for each exponent. Thus there are $3 \cdot 3 \cdot 3 = 27$ factors.

Individual Round — Algebra

(1) Find all real solutions of $|x-1|-4=2$.

$$\begin{aligned} |x-1|-4 &= 2 \\ |x-1| &= 6 \\ x-1 &= 6 \quad \text{or} \quad -6 \end{aligned}$$

There are 2 solutions: 7 and -5.

(2) Find all real numbers x, such that

$$2^{x^2-9x-52}=1.$$

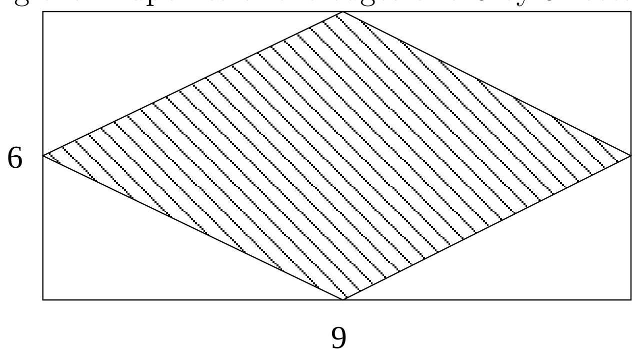
A power of 2 is 1, only if the exponent is zero. To find when the exponent is zero, we factor $x^2-9x-52$ as $(x-13)(x+4)$. The zeros correspond to these factors being zero. There are two answers: 13 and -4.

(3) Find a given that 5 is the remainder when $2x^3+x^2-ax-1$ is divided by $x-2$.

By synthetic division we find that $(2x^3+x^2-ax-1)$ when divide by $(x-2)$ is $2x^2+5x+(10-a)$ and the remainder is $2(10-a)-1$. Set the remainder equal to 5 and solve for a to find the solution $(a=7)$.

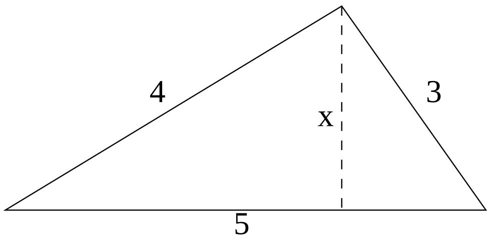
Individual Round — Geometry

(1) Find the area of the quadrilateral (shaded in the figure below) formed by joining the midpoints of the edges of a 6 by 9 rectangle.



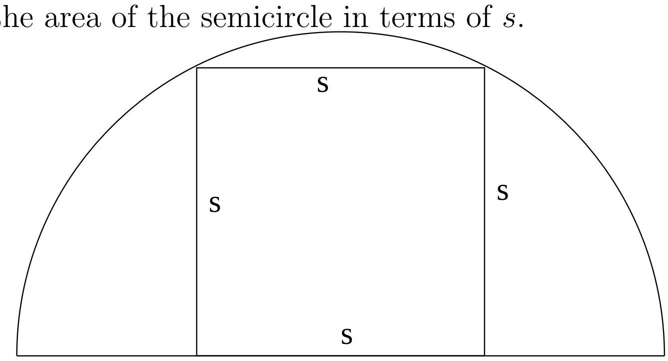
The diamond has half the area of the 6 by 9 rectangle. Its area is 27 square units.

(2) Find the altitude, x, of the triangle below.



The area of this triangle may be computed in two different ways: $A = \frac{1}{2} \cdot 3 \cdot 4$ and $A = \frac{1}{2} \cdot 5 \cdot x$. Equating these and solving for x gives $x = 12/5$.

(3) A square of side length s is inscribed in a semicircle as shown below. Find the area of the semicircle in terms of s.



The radius of the semicircle can be found by considering the line joining the midpoint of the bottom of the square to one of the upper corners. The radius is $\sqrt{(s/2)^2 + s^2}$ or $\sqrt{5s^2/4}$. The area of the semicircle is $A = \frac{1}{2}\pi 5s^2/4$ or $\frac{5}{8}\pi s^2$.

Team Round — CAPT

Bill and Melinda got a digital camera as a gift when their baby was born. They know they'll be taking lots of pictures and they'll need to make a lot of copies for family and friends.

Bill finds a service online from Kodak where 4 by 6 inch prints cost 15 cents each and shipping is $2.00 per order of up to 100.

Melinda thinks they should get their own photo printer. She finds a printer that costs $199.95 and the best deal on photo paper and an ink cartridge is a kit costing $36.99 that will yield 160 prints.

(1) What is the least amount they could spend to get 1000 prints under Bill's plan?

> If he always waits to order 100 prints at a time the cost is just $.02 per print so he will pay $.17 for each print, thus $170.00.

(2) How much would 1000 prints cost if instead they follow Melinda's plan?

> You could answer this several ways. Melinda would need to buy 7 "kits" in order to print 1000 photos, which gives a total cost of $458.88. Alternatively you can calculate on a per print basis using either $36.99/160 = $.2311875 or a rounded value of $.23 per print for the supplies. So $431.1375 and $429.95 are reasonable answers.

(3) Note that shipping costs $.02 per print, and the expense of photo paper and ink averages $.23 per print. Give formulas for the cost of x prints, calculated on a per print basis, for both schemes.

> Let's call the two functions $B(x)$ and $M(x)$.

$$B(x) = .17x \qquad \text{and} \qquad M(x) = 199.95 + .23x$$

(4) After much argument, Bill finally admits that the added convenience of having their own printer is worth something. He tells Melinda that if the overall cost of buying the printer and supplies is no more than double the cost of the online service they can go with her idea. Of course this calculation depends on how many prints they will need... What is the least number of paper and ink cartridge kits that they would need to go through in order to justify buying the printer?

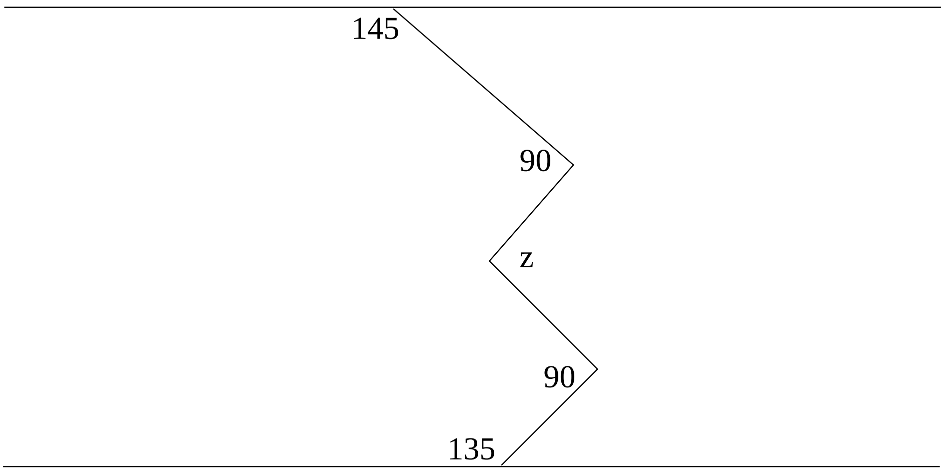
You need to find where $2 \cdot B(x)$ first exceeds $M(x)$, and this can be calculated in several ways based on your assumptions. Possible answers are 1817.72, 1837.56 and a few others. All of these lead to the same final answer of 12 kits.

Team Round — General

(1) The number **2006** is a 4 digit number having 2 zeros and two non-zero even digits. How many 4 digit numbers besides 2006 can also be described this way?

The thousands place digit can't be 0, otherwise we wouldn't truly have a four digit number. There are then 3 possible places to put the other non-zero digit. The possible forms of the numbers are XX00, X0X0 and X00X, where, the X's must be chosen from $\{2, 4, 6, 8\}$. There are 4 possible values for each X, so there are 16 numbers of each of the 3 forms; a total of 48 numbers. So the correct answer is 47.

(2) Find the angle z in the diagram below. (The lines at the top and bottom of the diagram are parallel, and the angle measures are in degrees.)



The answer is $100°$. There are many different ways to justify this, the easiest is to draw lines parallel to the top and bottom lines through each angle and note supplementary angles.

(3) Find x and y if

$$2^{x+y} = 128 \qquad \text{and} \qquad 32^{x-y} = 512$$

Rewrite the first equation as $2^{x+y} = 2^7$, and the second equation as $2^{5(x-y)} = 2^9$. We get the system of equations

$$x + y = 7 \qquad \text{and} \qquad 5(x - y) = 9$$

You can find the answer either by substitution or elimination ($x = 22/5$ and $y = 13/5$).

(4) How many different triangles can be constructed by choosing three vertices from among the corners of a unit cube?

> Every set of three vertices chosen from among the eight corners will be a triangle (just note that no three of the cube's corners fall in a line). There are 8 choices for the first vertex, 7 choices for the second and 6 for the last, thus $8 \cdot 7 \cdot 6 = 336$ — but this would overcount the triangles by a factor of 6 since ABC, ACB, BAC, BCA, CAB, and CBA would all get counted separately in the above calculation but they are all, obviously, the same triangle.

How many of these triangles have different shapes. That is, how many non-congruent triangles are there?

> There are three non-congruent triangles. Twenty-four triangles appear in the faces of the cube (4 in each of the 6 faces) that have sides of lengths $(1,1,\sqrt{2})$. There are also 24 triangles which have sides of length $(1, \sqrt{2}, \sqrt{3})$. Finally, there are 8 equilateral triangles whose sides are all $\sqrt{3}$.