

## Individual Round — Arithmetic

- (1) A stack of 100 nickels is 6.25 inches high. To the nearest \$.01, how much would a stack of nickels 8 feet high be worth?

8 feet =  $8 \cdot 12$  inches. Dividing 96 inches by 6.25 and multiplying by 100 gives the number of nickels in the stack (1536). The value of the stack is then  $$.05 \cdot 1536 = \$76.80$

- (2) Continue the pattern:

$$1^3 = 1^2 - 0^2$$

$$2^3 = 3^2 - 1^2$$

$$3^3 = 6^2 - 3^2$$

$$4^3 = \underline{10^2 - 6^2}$$

$$5^3 = \underline{15^2 - 10^2}$$

- (3) Determine the largest prime divisor of  $87! + 88!$  (The exclamation point indicates the *factorial* of a number – the product of all the integers from 1 to that number. For example,  $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$ .)

$$87! + 88! = 87! + 88 \cdot 87! = 89 \cdot 87!$$

All the primes dividing  $87!$  are less than 87, so the answer is 89.

## Individual Round — Algebra

- (1) The sum of three numbers is 98. The first number is  $2/3$  of the second and the second is  $5/8$  of the third. What is the second number?

$$x + y + z = 98 \quad \text{and} \quad x = 2/3y \quad \text{and} \quad y = 5/8z$$

Everything can be expressed in terms of  $y$ .

$$z = 8/5 \cdot y \quad \text{and} \quad x = 2/3 \cdot y \quad \text{and of course} \quad y = y.$$

Thus the first equation becomes  $2/3 \cdot y + y + 8/5 \cdot y = 98$ .

Which yields  $y \cdot (2/3 + 1 + 8/5) = 98$ . Finally, simplifying this

$$\text{should give } y = 98 \cdot \frac{15}{42} = 30.$$

- (2) Find a value of  $x$  that satisfies the following:

$$8^{10x} = 32^{7x-5}$$

Re-express the bases of these exponential expressions as powers of 2.

$$(2^3)^{10x} = (2^5)^{7x-5}$$

The laws of exponents then show that  $2^{30x} = 2^{5(7x-5)}$ . The expressions in the exponents must be equal.

$$30x = 35x - 25$$

Which has  $x = 5$  as a solution.

- (3) A number  $x$  satisfies the equation  $x + \frac{1}{x} = 4$ . Find the value of  $x^2 + \frac{1}{x^2}$ .

Solution I:  $(x + 1/x)^2 = 4^2$  so  $x^2 + 2 \cdot x \cdot 1/x + (1/x)^2 = 16$ . You can simplify this to get  $x^2 + 2 + 1/x^2 = 16$ . Which leads to  $x^2 + 1/x^2 = 14$ .

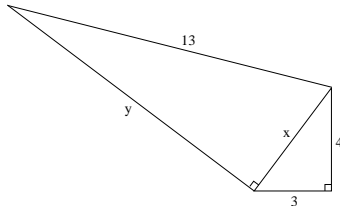
Solution II: Multiply  $x + 1/x = 4$  by  $x$  to get a quadratic equation.

$$x^2 + 1 = 4x \quad \text{or} \quad x^2 - 4x + 1 = 0$$

By the quadratic formula,  $x$  is either  $\frac{4 + \sqrt{12}}{2}$  or  $\frac{4 - \sqrt{12}}{2}$  (these may be simplified to  $2 + \sqrt{3}$  and  $2 - \sqrt{3}$ ). Both roots give 14 when you substitute them into  $x^2 + 1/x^2$ .

## Individual Round — Geometry

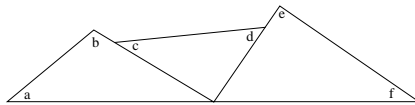
What is the length of the segment marked  $y$  in the following diagram?



If you know the standard right triangles, 3-4-5 and 5-12-13, you get  $y = 12$  immediately. Otherwise, use the Pythagorean theorem to find  $x = \sqrt{3^2 + 4^2} = 5$ . Then use the Pythagorean theorem a second time to find  $y$ :  $13^2 = y^2 + 5^2$  so  $y^2 = 169 - 25 = 144$ , which leads to  $y = 12$ .

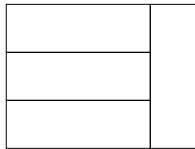
Find the value in degrees of the sum of the six angles indicated in the diagram.

$$a + b + c + d + e + f = \underline{360^\circ}$$



Call the other 3 angles  $x$ ,  $y$  and  $z$ . Because the angles on a triangle add up to  $180^\circ$  we get that  $a + b + c + d + e + f + x + y + z = 3 \cdot 180^\circ$ . On the other hand, together  $x$ ,  $y$  and  $z$  make a  $180^\circ$  angle so  $a + b + c + d + e + f = 2 \cdot 180^\circ$  or  $360^\circ$ .

Four congruent rectangles are arranged as shown below to form a large rectangle. The area of the large rectangle formed is 768. What is the area of a square having the same perimeter as the large rectangle?



Let the short side be  $x$  and the longer side be  $y$ . From the diagram we can see that  $y = 3x$ , and since the area of 4 copies of the rectangle is 768, we get  $xy = 192$  or  $x \cdot 3x = 192$  which gives  $x^2 = 64$  and finally  $x = 8$ . Return to the diagram to see that the perimeter of the big rectangle can be divided into 14 segments of length  $x$ , so the perimeter is 112. A square having perimeter 112 will have sides of length 28, which gives an area of 784.

## Team Round — CAPT

A consultant is hired to analyze the pricing of one of the product lines at HSM Corporation. The work of a previous consultant is made available which shows that the relationship between  $p$  (the price, which is determined by HSM) and  $q$  (the quantity sold in a month, which is determined by market forces) is

$$q = 20 \cdot (500 - p)$$

According to this model, how many units should HSM expect to “sell” if they are giving them away for free? 10,000

Just evaluate the equation above at  $p = 0$ .

If HSM sets the price too high demand for this product will drop. At what price would the quantity sold ( $q$ ) be zero? 500

The formula for  $q$  has a root at  $p = 500$ .

The revenue generated by this item is the product of the price set ( $p$ ) and the quantity sold ( $q$ ). Find the ideal price – the price at which the most revenue is generated. 250

$$q \cdot p = (20 \cdot (500 - p)) \cdot p = 20 \cdot (500p - p^2)$$

That is a quadratic expression with roots at 0 and 500. The highest point on the curve falls exactly midway between the roots at  $p = 250$ .

Some additional data is brought to the consultant: During January of 2005 the price of the product was set at \$260.00 and the quantity sold in January was 5800 units. During February of 2005 the price of the product was reduced to \$220.00 which stimulated sales, the quantity sold in February was 6600 units. Using this new data to determine an updated model for the relationship between price ( $p$ ) and quantity sold ( $q$ ). What is the ideal price according to your new model? 275

The new formula for  $q$  should be

$$q = 20 \cdot (550 - p).$$

This changes the expression for the revenue into a quadratic with roots at 0 and 550. The ideal price again falls midway between 0 and 550 or  $p = 275$ .

## Team Round — General

- (1) What is the sum of the numbers between 1 and 300 that are divisible by either 11 or 13 (or both)?

There are roughly  $300/11$  numbers from 1 to 300 that are divisible by 11. Since this must be a whole number, the actual value is 27. There are about  $300/13$  numbers in that range that are divisible by 13, but again, we must round down to get the exact number (23). The sum of the numbers divisible by 11 is:

$$11 + 22 + \dots + 297 = 11 \cdot (1 + 2 + \dots + 27) = 11 \cdot \frac{27 \cdot 28}{2} = 4158.$$

The sum of the numbers divisible by 13 is:

$$13 + 26 + \dots + 299 = 13 \cdot (1 + 2 + \dots + 23) = 13 \cdot \frac{23 \cdot 24}{2} = 3588.$$

If you gave 7746 as the answer it means you forgot the two numbers (143 and 286) which are divisible by both 11 and 13, these have been counted twice above. The correct value is 7317.

- (2) A rectangular patio is made up of **2005** square tiles. There are some red tiles and some that are white.

The tiles on the border, one tile in width, are all red and all the interior tiles are white. How many white tiles are there?

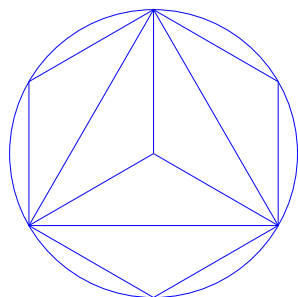
The only ways to factor 2005 are  $1 \cdot 2005$  and  $5 \cdot 401$ . The first factorization wouldn't give any interior tiles and we know there are some, so the patio must be 5 by 401. The white tiles form a rectangle that is 2 smaller in each direction so the answer is  $3 \cdot 399$  or 1197.

- (3) Find three consecutive odd numbers such that the sum of their squares is a four digit number with all four digits identical.

A quick way to do this one on a graphing calculator is to enter  $y = (2x+1)^2 + (2x+3)^2 + (2x+5)^2$  and use the "table" mode. Just scan down the list until you find a number with 4 identical digits.

It turns out that  $5555 = 41^2 + 43^2 + 45^2$ .

- (4) An equilateral triangle and a regular hexagon are inscribed in a circle. What is the ratio of their areas?



$$\frac{\text{area of triangle}}{\text{area of hexagon}} = \frac{1}{2}$$

It's possible to cut a second copy of the triangle into 3 pieces and place both the original and the pieces of the copy inside the hexagon.